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LOCAL AND RENORMALIZABLE ACTION FROM THE GRIBOV HORIZON

Daniel ZWANZIGER*

Physics Department, New York University, New York, NY 10003, USA

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We derive a local, renormalizable action for non-abelian gauge theories, which expresses the restriction of the domain of the functional integral to the interior of the Gribov horizon, and show that the divergences may be absorbed by field and coupling constant renormalization. The condition that the euclidean functional integral extends up to the boundary of the classical configuration space provides an absolute normalization of the gauge field and eliminates the perturbative coupling constant g^2 in favor of a dimensionful parameter γ. In $D = 4$ dimensions, with dimensional regularization, the coupling constant $g^2(\epsilon)$ is found, in zeroth order, to vanish with $\epsilon = 4 - D$, in accordance with asymptotic freedom. In consequence of the restriction of the classical configuration space, the poles of the gluon propagator are shifted, in zeroth order, to an unphysical location at $p^2 = \pm i\gamma^{1/2}$, but the glueball channel contains a physical cut with positive spectral function.

1. Introduction

Since the work of Gribov, it is known that the restriction of the euclidean functional integral to a fundamental modular region has dynamical implications [1]. He proposed that the functional integral over the connection A be restricted to the region Ω which is defined by the conditions (1) that A be transverse

$$\partial \cdot A = 0, \tag{1.1}$$

and (2) that the Faddeev–Popov operator $K[A]$ be positive

$$K[A] \equiv -\partial^2 - A \cdot \partial \geqslant 0. \tag{1.2}$$

[Here $A = A_\mu^b(x)$ acts in the adjoint representation

$$(A_\mu u)^a(x) \equiv f^{abc} A_\mu^b(x) u(x)^c, \tag{1.3}$$

and f^{abc} are the structure constants of the Lie algebra of a compact, semi-simple

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Lie group.] However it has taken some time to implement this concretely. An early result [2] was the proof that Ω enjoys three simple properties: (1) Ω is convex, (2) Ω is bounded in every direction and (3) every gauge orbit passes through Ω at least once. Recently it has been shown [3] that in second-order perturbation theory, the boundary of Ω, known as the Gribov horizon, is an ellipsoid described as follows. Let the euclidean connection in D dimensions $A^c_\mu(x)$, defined on a periodic box, have the Fourier expansion

$$A^c_\mu(x) = V^{-1/2} \sum_k a^c_\mu(k) \exp(ik \cdot x), \tag{1.4}$$

so transversality is expressed by $k \cdot a(k) = 0$. Then, in second-order perturbation theory and for large volumes, the boundary of Ω is the ellipsoid

$$V^{-1} \sum_{k,\mu,b} \left|a^b_\mu(k)\right|^2 \Big/ k^2 = MD/C, \tag{1.5}$$

where M is the dimension of the gauge group, and $C > 0$ is the value of the Casimir operator in the adjoint representation defined by $f^{abc} f^{cda} = -\delta^{bd} C$, where f^{abc} are the structure constants of the gauge group. [For gauge group SU(N), with conventional normalization, one has $M = N^2 - 1$, and $C = N$.] It was also found [3] that, in lowest order, the restriction of the functional integral to the interior of this ellipsoid leads to a zero-order gluon propagator of the form

$$\left(\delta_{\mu\nu} - k_\mu k_\nu / k^2\right) \delta^{bc} \left(k^2 + \gamma/k^2\right)^{-1}, \tag{1.6}$$

which vanishes at low momentum like k^2. More recently [4] it has been shown rigorously that Ω is contained within the ellipsoid

$$V^{-1} \sum_{k,\mu,b} \left|a^b_\mu(k)\right|^2 \Big/ k^2 = \text{const}\, D, \tag{1.7}$$

where the sum extends over all $k \neq 0$, and D is the dimension of euclidean space-time. (For the gauge group SU(2), we have const = 60.) By taking the vacuum expectation of this inequality it follows that, in the infinite-volume limit, the gluon propagator in momentum space $g(k)$ satisfies

$$\int \mathrm{d}^D k\, g(k) \Big/ k^2 \leqslant \text{const}. \tag{1.8}$$

This bound contradicts the ultraviolet behavior predicted by the perturbative renormalization group in 4-space-time dimensions [4]. [Here $g(k)$ is the scalar function which is obtained after removing trivial kinematic Lorentz and color indices from the gluon propagator.] This has provided a strong motivation for the present work.

The principle result obtained here is the derivation of a local action that is renormalizable by power counting, and which expresses the restriction of the functional integral to the interior of Ω. We also obtain a normalization condition for the connection A, which expresses the condition that the domain of the euclidean functional integral which, in accordance with Feynman's ideas, is a sum over classical paths, extends up to the boundary of the classical configuration space. We also discuss a possible mechanism for the confinement of gluons [5]*.

In the present work, we use interchangeably the terms "fundamental modular region" and "classical configuration space". This is the image of the gauge orbit space in A-space under a satisfactory gauge fixing. As noted above, property (3), it is known that every gauge orbit passes through Ω, the region defined by conditions (1.1) and (1.2), which shows that the fundamental modular region is contained in Ω. In the present work, we take as a working hypothesis that the fundamental modular region is Ω itself, rather than some proper subset of it, although no proof of this exists, and we freely refer to Ω as the classical configuration space. Should it turn out that some proper subset of Ω is the classical configuration space, rather than Ω itself, then the domain of the functional integral is more restricted, and the physical consequences more drastic than what we find here. Note that because Ω is convex [property (1) above], it is topologically trivial, whereas it is known that the gauge-orbit space is in fact topologically non-trivial and that this is in fact the origin of anomalies**. However the topological non-triviality may be accounted for by identifying points on the boundary of Ω. In fact, one may show that if Ω is the classical configuration space, then its boundary is locally fibered into curves that each lie on a single gauge orbit [7].

The reader may find it helpful to turn directly to the concluding section, where the results of the present article are summarized and the final formulas are assembled, before turning to their derivation in the remaining sections.

In sect. 2 we analyze a simple model, in which the Faddeev–Popov measure is replaced by a gaussian free field, and the Gribov horizon is an ellipsoid, and show that the cut-off of the functional integral at the Gribov horizon may be replaced by a Boltzmann weight. In sect. 3 we calculate $\lambda[A]$, the least eigenvalue of the Faddeev–Popov operator, in the limit of large volumes, and obtain explicit formu-

* For a recent discussion of gluon confinement which is not based on the mechanism proposed here, and for further references, see ref. [5]. There, the authors, and their predecessors, obtain a gluon propagator which behaves at low momentum like $1/k^4$. This behavior, like the free propagator in 4 dimensions violates the rigorous bound (1.8). In the present work, which is based on another action that incorporates the restriction to the interior of the Gribov horizon, it is found that the gluon propagator behaves at low momentum like k^2 in zeroth order, and this suppression is expected to persist to higher order.

** For a review of this topic see ref. [6].

las, eqs. (3.14) and (3.16), for the location of the Gribov horizon at large volume. In sect. 4 we derive the euclidean functional integral of the gauge theory, in the form of a positive but non-local measure in A-space, and we obtain the absolute normalization condition, which expresses the fact that the functional integral extends up to the boundary of the classical fundamental modular region. In sect. 5 we show that the euclidean functional integral may be expressed in terms of a local renormalizable action involving ghost fields, and we show that the divergences encountered in perturbation theory may be absorbed in a renormalization of the fields and coupling constants. We discuss the physical interpretation of the zero-order theory, and a possible mechanism for the confinement of gluons. In the appendix, the correlation function of two small Wilson loops is calculated in zeroth-order perturbation theory, with a result that is analyzed in sect. 5.

Our conventions are as follows. We consider connections $A_\mu^c(x)$ on a euclidean base manifold which is a periodic box of edge L, whose points are parametrized by coordinates x_μ, $\mu = 1, \ldots, D$, which vary in the interval $0 \leqslant x_\mu \leqslant L$. The index $c = 1, \ldots, M$, labels components in the adjoint representation of the Lie algebra of a compact semi-simple Lie group. Here M is the order of the group which, for $SU(N)$, is $M = N^2 - 1$. We call this the "color index", without prejudice to other physical interpretations of the gauge theory, and similarly we shall call the 2-point correlation function of the connection A the "gluon propagator". We restrict our attention to the sector with zero instanton number (Pontrjagin index), so the bundle is trivial and $A(x)$ is a globally defined function on the base manifold. Henceforth, A shall always represent a transverse connection, i.e. one satisfying eq. (1.1).

2. A simple model in which the Gribov boundary perturbs gluons into shadow particles

As a guide, let us consider the simple model in which the Faddeev–Popov action is replaced by the free-field action

$$S_0[A] = \sum_k |a(k)|^2 q(k^2), \tag{2.1}$$

and the Gribov horizon $\partial\Omega$ is the ellipsoid in A-space

$$Q[A] \equiv V^{-1} \sum_k |a(k)|^2 p(k^2) = c, \tag{2.2}$$

where the sum extends over all $k \neq 0$. Here $a(k)$ is the Fourier coefficient of a scalar field $A(x)$ in a periodic box of euclidean volume $V = L^D$,

$$A(x) = V^{-1/2} \sum_k a(k) \exp(ik \cdot x), \tag{2.3}$$

and $k_\mu = 2\pi n_\mu / L$, where the n_μ are integers. For the moment we leave $p(k^2)$ and $q(k^2)$ unspecified, apart from the requirement that they be positive, although we shall ultimately be interested in the case where $q(k^2)$ is the inverse of the massless free-field propagator

$$q(k^2) = k^2 \tag{2.4}$$

and $p(k^2)$, which is the inverse of the square of a semi-major axis of the ellipsoid is given by

$$p(k^2) = 1/k^2 . \tag{2.5}$$

Our model is defined by the measure

$$\mathrm{d}\mu_c = \mathrm{d}A \exp(-S_0)\theta(c - Q[A]), \tag{2.6a}$$

corresponding to the partition function

$$Z_c \equiv \int \mathrm{d}A \exp\left[-\sum_k |a(k)|^2 q(k^2)\right] \theta(c - Q[A]). \tag{2.6b}$$

It is instructive to calculate the mean and variance of the random variable $Q[A]$ in the probability measure $\mathrm{d}\mu = \mathrm{d}A \exp(-S_0[A])$. Since it is gaussian we immediately obtain

$$\langle Q[A] \rangle = V^{-1} \sum_k p(k^2)/q(k^2),$$

$$\langle Q^2[A] \rangle = \langle Q[A] \rangle^2 + V^{-2} 2 \sum_k p^2(k^2)/q^2(k^2).$$

At large volumes these approach

$$\langle Q[A] \rangle = c_0,$$

where

$$c_0 \equiv (2\pi)^{-D} \int \mathrm{d}^D k \, p(k^2)/q(k^2), \tag{2.7}$$

and

$$\langle Q^2[A] \rangle = \langle Q[A] \rangle^2 + V^{-1} 2 (2\pi)^{-D} \int \mathrm{d}^D k \, p^2(k^2)/q^2(k^2).$$

This gives

$$\langle Q^2[A] \rangle = \langle Q[A] \rangle^2 \tag{2.8}$$

for $V \to \infty$, and thus the variance of $Q[A]$ is seen to vanish in the infinite-volume limit.

We conclude that in the gaussian free-field measure, $\mathrm{d}A \exp(-S_0)$, the random variable $Q[A]$ has a δ-function distribution, with mean $\langle Q \rangle = c_0$. If in our model, defined by $\mathrm{d}\mu_c = \mathrm{d}A \exp(-S_0)\, \theta(c-Q)$, the constant c is greater than c_0, the restriction imposed by the θ-function $\theta(c - Q[A])$ is vacuous, and leads to the trivial result, $\mathrm{d}\mu_c = \mathrm{d}A \exp(-S_0)$. We restrict our considerations henceforth to the non-trivial case

$$c < c_0 . \tag{2.9}$$

(Note that if $p(k^2)$ and $q(k^2)$ are as in eqs. (2.4) and (2.5), then

$$c_0 = \int \mathrm{d}^D k \, (k^2)^{-2}$$

diverges at low k for euclidean space-time dimension $D \leqslant 4$, and at high k for $D \geqslant 4$, so the relation $c < c_0$ holds for all finite c, and the non-trivial case holds.)

A difficulty arizes in the non-trivial case, $c < c_0$, for it appears that the measure $\mathrm{d}\mu_c$ or the partition function Z_c vanishes. This vanishing however turns out to be a problem of the overall normalization of Z_c, which we shall overcome by renormalizing it. Let us consider how this may be done. The apparent vanishing of $\mathrm{d}\mu_c = \mathrm{d}A \exp(-S_0)\, \theta(c-Q)$ means that the support of the gaussian measure $\mathrm{d}A \exp(-S_0)$ lies outside the ellipsoid $Q[A] = c_0$. We analyze this case in more detail. Suppose a change of variables is made from $a(k)$ to

$$y_k \equiv a(k)\, p^{1/2}(k) ,$$

so the ellipsoid, $Q[A] = c$, in A-space is mapped into the sphere in y-space

$$y^2 \equiv \sum_k y_k^2 = Vc .$$

The integral over y extends over the volume of the ball in y-space which is bounded at the radius $r \equiv (y^2)^{1/2} = R \equiv (Vc)^{1/2}$. Let the functional integral be regularized by a cut-off in momentum space, so the dimension of A-space or y-space is the finite but large number N. In a euclidean space of dimension N, the volume element in the radial variable $r \equiv (y^2)^{1/2}$ is $r^{N-1}\mathrm{d}r$, and as N grows without limit, the volume of the ball of radius R, becomes concentrated at its surface, the sphere of radius R. (The actual ellipsoidal or spherical shape of our model Gribov horizon is essential to the argument; it is not true, for example, that the volume of a hypercube of high dimension N gets concentrated at its surface.) Thus, since A-space is infinite dimensional, we could replace the ball of radius R by the sphere of radius R. In other words, we could make the substitution

$$dA\exp(-S_0)\,\theta(c-Q[A]) \to dA\exp(-S_0)\,\delta(c-Q[A]). \qquad (2.10a)$$

This also appears to vanish if $c < c_0$. However the regularized functional integral gets a small, but non-zero, contribution at $y^2 = Vc$, which is in the tail of the gaussian $\exp(-S_0)$. This may be small, but it is the leading contribution in the case $c < c_0$, and this allows us to renormalize it. (In the opposite case, $c > c_0$, the leading contribution to $dA\exp(-S_0)$ lies in the interior of the ball, so the limit of integration at $r = R$ becomes irrelevant as the cut-off in momentum space is removed.)

Let us temporarily assume that the equivalence of the microcanonical and canonical ensembles in classical statistical mechanics is valid here, so that it is correct to replace the δ-function by the Boltzmann factor

$$dA\exp(-S_0)\,\delta(c-Q[A]) \to dA\exp(-S_0)\exp(-\gamma VQ[A]) \qquad (2.10b)$$

and the measure $dA\exp(-S_0)\,\theta(c-Q)$ which defines our model is replaced by

$$d\mu_\gamma \equiv dA\exp(-S_0[A]-\gamma S_1[A]), \qquad (2.11a)$$

corresponding to the partition function

$$Z_\gamma = N\int dA\exp(-S_0[A]-\gamma S_1[A]), \qquad (2.11b)$$

where

$$S_1[A] \equiv VQ[A] = \sum_k |a(k)|^2 p(k^2). \qquad (2.12)$$

Here γ is a statistical mechanical parameter whose value is determined by the condition

$$c(\gamma) \equiv \langle Q[A]\rangle_\gamma = c, \qquad (2.13)$$

where the expectation value is taken with respect to the measure $d\mu_\gamma$. Since this measure is also gaussian, one finds without difficulty

$$c(\gamma) = V^{-1}\sum_k p(k^2)\left[q(k^2)+\gamma p(k^2)\right]^{-1} = c,$$

or, in the infinite-volume limit,

$$c(\gamma) = (2\pi)^{-D}\int d^Dk\, p(k^2)\left[q(k^2)+\gamma p(k^2)\right]^{-1} = c. \qquad (2.14)$$

Comparison with eq. (2.7) shows that for $c < c_0$, which is the case at hand, there is always a solution with

$$\gamma > 0. \tag{2.15}$$

To prove the equivalence of the canonical and microcanonical ensembles, we observe that $Q[A]$ has zero variance as in eq. (2.8),

$$\langle Q^2 \rangle_\gamma = \langle Q \rangle_\gamma^2, \tag{2.16}$$

as follows by repeating the calculation described above, leading to eq. (2.8), for the gaussian measure $\mathrm{d}\mu_\gamma$ instead of $\mathrm{d}\mu_0$. Thus the random variable $Q[A]$ has a δ-function distribution in the Boltzmann distribution $\mathrm{d}\mu_\gamma$, and the microcanonical and canonical distributions are indeed equivalent in the present case.

What have we learned from the present exercise? Our final result is a gaussian field with propagator (covariance)

$$\left[q(k^2) + \gamma p(k^2)\right]^{-1}. \tag{2.17}$$

The cut-off at the Gribov horizon introduces the term proportional to γ which may be regarded as a perturbation of the unperturbed propagator $1/q(k^2)$. Let us take as a first example

$$q(k^2) = k^2 + m^2, \tag{2.18}$$

which describes a free field of mass m, and for $p(k^2)$ we take

$$p(k^2) = (k^2 + m^2)^{-1}. \tag{2.19}$$

The perturbation results in the propagator

$$\left[k^2 + m^2 + \gamma(k^2 + m^2)^{-1}\right]^{-1} = \tfrac{1}{2}\left[(k^2 + m^2 + i\gamma^{1/2})^{-1} + (k^2 + m^2 - i\gamma^{1/2})^{-1}\right]. \tag{2.20}$$

This propagator describes a resonance of mass m and width $\gamma^{1/2}$. Thus, the perturbation has changed a stable particle into an unstable one. With this choice for $q(k^2)$ and $p(k^2)$, γ is determined by the equation

$$c(\gamma) = (2\pi)^{-D} \int \mathrm{d}^D k \left[(k^2 + m^2)^2 + \gamma\right]^{-1} = c. \tag{2.21}$$

Because of the complex poles, the propagator (2.20) does *not* satisfy the Kallen–Lehmann representation, so the present model is only a euclidean model. It does not describe a physical model in Minkowski space. We will return to this important question in the context of gauge theories in the last section.

The massless case $m=0$ is of the greatest interest to us, for it agrees with the zeroth-order perturbative approximation to the gauge theory considered in sect. 4. In this case the propagator is

$$\left[k^2+\gamma(k^2)^{-1}\right]^{-1}=\tfrac{1}{2}\left[\left(k^2+i\gamma^{1/2}\right)^{-1}+\left(k^2-i\gamma^{1/2}\right)^{-1}\right]. \tag{2.22}$$

Thus, in zeroth-order perturbation theory, the gluon may be described as some kind of a shadow particle which is a resonance at zero mass but of finite width. The thermodynamic parameter γ is determined by

$$c(\gamma)=(2\pi)^{-D}\int \mathrm{d}^D k\left[(k^2)^2+\gamma\right]^{-1}=c, \tag{2.23}$$

or

$$c=c(\gamma)=S_{D-1}(2\pi)^{-D}\tfrac{1}{4}\pi\left[\sin\left(\tfrac{1}{4}\pi D\right)\right]^{-1}\gamma^{-1+D/4}, \tag{2.24a}$$

where S_{D-1} is the area of the $(D-1)$-dimensional sphere

$$S_{D-1}=2\pi^{D/2}/\Gamma\left(\tfrac{1}{2}D\right), \tag{2.24b}$$

and we have used the identity (A.24).

The reader will have noticed that the integrals (2.21) or (2.23) converge only if

$$D<4, \tag{2.25}$$

so our renormalization is successful and the model is defined only in these euclidean space-time dimensions. We shall study the limit $D\to 4$ in the context of the gauge theory, to which we turn next.

3. Gribov horizon at large volumes

Let us consider the measure appropriate to gauge theories,

$$\mathrm{d}\mu\equiv \mathrm{d}A\,\delta(\partial\cdot A)\exp\left(-g^{-2}S_{\mathrm{cl}}[A]\right)\det(K[A])\,\theta(\lambda[A]), \tag{3.1a}$$

where

$$S_{\mathrm{cl}}[A]=(1/4)\sum_{\mu,\nu,a}\int \mathrm{d}^D x\left(F^a_{\mu,\nu}\right)^2(x), \tag{3.1b}$$

$$F^a_{\mu,\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+f^{abc}A^b_\mu A^c_\nu, \tag{3.1c}$$

$$K[A]=-\partial^2-A\cdot\partial, \tag{3.1d}$$

and $\lambda[A]$ is the lowest eigenvalue of the Faddeev–Popov operator $K[A]$. We wish to follow the method used in the simple model. To do so we need a formula for $\lambda[A]$. It will turn out that in the infinite-volume limit, the resummation of the perturbation series for $\lambda[A]$ is possible.

We write in an obvious notation

$$K[A] = K_0 + K_1[A] = -\partial^2 - A\cdot\partial\,. \tag{3.2}$$

Both K_0 and $K_1[A]$ possess a trivial null-space N_0, consisting of constant color vectors u^{c}. This null-space obviously does not contain any negative eigenvalue for any A. Therefore, to calculate the boundary of Ω, which is where $K[A]$ acquires its first negative eigenvalue, we shall consider K_0 and $K_1[A]$ to act on the subspace orthogonal to this trivial null-space, and we shall perturb around the next-lowest-lying eigenspace of $K_0 = -\partial^2$, which belongs to the eigenvalue $\lambda_0 = (2\pi/L)^2$. The corresponding eigenvectors may be labeled by $|k_0, b\rangle$, where b is a color index, $b = 1,\ldots, M$, and k_0 is a momentum vector satisfying

$$k_0^2 = \lambda_0 = (2\pi/L)^2\,. \tag{3.3}$$

There are $2D$ such momentum vectors in D-dimensional euclidean space-time (which point along the principal axes in positive and negative directions), so there are $2DM$ degenerate states in this level, and we must use degenerate perturbation theory.

We shall develop the formalism of degenerate perturbation theory in a form which is convenient for our purposes. Let P_0 be the projector onto the eigenspace H_0 of K_0 which belongs to the eigenvalue λ_0,

$$P_0 = \sum_{k_0,\, b} |k_0, b\rangle\langle k_0, b|\,. \tag{3.4}$$

Here and below, the summation over k_0 is restricted to momenta that satisfy $k_0^2 = (2\pi/L)^2$. We write the eigenspace problem in the form

$$KS = S\kappa\,, \tag{3.5a}$$

where

$$\kappa P_0 = P_0\kappa = \kappa\,, \tag{3.5b}$$

and S is an operator that satisfies

$$SP_0 = S\,. \tag{3.5c}$$

The significance of S is that it maps the eigenspace H_0 of K_0 which belongs to the eigenvalue $\lambda_0 = k_0^2$ and which is spanned by the $|k_0, b\rangle$, onto the space spanned by the eigenvectors of K that these states evolve into as the perturbation is turned on. Equations (3.5b) and (3.5c) state that S and κ annihilate the subspace orthogonal to H_0. The operator κ may be represented as a $2MD \times 2MD$ matrix that acts in H_0. Both S and κ may be determined perturbatively. The principle characteristic of degenerate perturbation theory is that the exact eigenvectors cannot be determined perturbatively, but must be obtained by diagonalizing κ by a large transformation that is not perturbative. In the present case, this means diagonalizing a matrix of dimension $2DM$. We write

$$S = \sum_n S_n, \qquad \kappa = \sum_n \kappa_n, \tag{3.6}$$

where the sum extends from 0 to ∞. On substituting this into eq. (3.5a) and equating coefficients of like order we obtain

$$K_0 S_0 = S_0 \kappa_0, \tag{3.7a}$$

$$K_0 S_1 + K_1 S_0 = S_1 \kappa_0 + S_0 \kappa_1, \tag{3.7b}$$

$$K_0 S_2 + K_1 S_1 = S_2 \kappa_0 + S_1 \kappa_1 + S_0 \kappa_2, \tag{3.7c}$$

$$K_0 S_3 + K_1 S_2 = S_3 \kappa_0 + S_2 \kappa_1 + S_1 \kappa_2 + S_0 \kappa_3, \tag{3.7d}$$

etc.

The first equation is solved by

$$S_0 = P_0, \qquad \kappa_0 = \lambda_0 P_0 = (2\pi/L)^2 P_0. \tag{3.8a, b}$$

To solve the remaining equations, we look for solutions that satisfy the normalization conditions

$$P_0 S_n = 0, \quad \text{for } n \geqslant 1. \tag{3.9a}$$

By eq. (3.5c) they also satisfy

$$S_n = S_n P_0. \tag{3.9b}$$

Upon multiplying eqs. (3.7b)–(3.7d) on the left by P_0 we obtain

$$\kappa_1 = P_0 K_1 P_0, \qquad \kappa_2 = P_0 K_1 S_1, \qquad \kappa_3 = P_0 K_1 S_2, \tag{3.10a, b, c}$$

etc. The S_n and κ_n are now uniquely determined recursively, and we obtain

$$S_1 = -[(I-P_0)/(K_0-\lambda_0)]K_1P_0,$$

$$S_2 = \{[(I-P_0)/(K_0-\lambda_0)]K_1\}^2P_0 - \left[(I-P_0)/(K_0-\lambda_0)^2\right]K_1P_0K_1P_0,$$

so

$$\kappa_1 = P_0K_1P_0,$$

$$\kappa_2 = -P_0K_1[(I-P_0)/(K_0-\lambda_0)]K_1P_0,$$

$$\kappa_3 = P_0K_1\{[(I-P_0)/(K_0-\lambda_0)]K_1\}^2P_0 - P_0K_1\left[(I-P_0)/(K_0-\lambda_0)^2\right]K_1P_0K_1P_0,$$

etc. In the basis $|k_0, b\rangle$ which spans H_0, the operators κ_n are represented by the matrices

$$\kappa_1 = \langle k_0, b|K_1|k_0', c\rangle, \tag{3.11a}$$

$$\kappa_2 = -\langle k_0, b|K_1[(I-P_0)/(K_0-\lambda_0)]K_1|k_0', c\rangle, \tag{3.11b}$$

$$\begin{aligned}\kappa_3 = {} & \langle k_0, b|K_1\{[(I-P_0)/(K_0-\lambda_0)]K_1\}^2|k_0', c\rangle \\ & -\langle k_0, b|K_1\left[(I-P_0)/(K_0-\lambda_0)^2\right]K_1P_0K_1|k_0', c\rangle,\end{aligned} \tag{3.11c}$$

etc., where $k_0^2 = k_0'^2 = (2\pi/L)^2$.

We wish to estimate these expressions in the limit of large euclidean volume $V = L^D$. For this purpose we use momentum-state expressions, namely eq. (3.4) for P_0 and

$$(I-P_0)/(K_0-\lambda_0) = \sum_{k,d} |k, d\rangle\left(k^2-k_0^2\right)^{-1}\langle k, d|,$$

where the sum on k is subject to the restriction $k^2 > k_0^2 = (2\pi/L)^2$. With $K_1 = -A\cdot\partial$, we then obtain in κ_n expressions of the type

$$\begin{aligned}\langle k_1, b|K_1|k_2, c\rangle &= -ik_{2\mu}\langle k_1, b|A_\mu|k_2, c\rangle \\ &= -ik_{1\mu}\langle k_1, b|A_\mu|k_2, c\rangle,\end{aligned}$$

where the last equality holds because A is transverse, $\partial\cdot A = 0$. In the limit of large volume $V = L^D$, momenta k_0, which satisfy $k_0^2 = (2\pi/L)^2$, will be negligible compared to momenta k which survive in the infinite-volume limit. Thus the last expression is of order $(2\pi/L)$ if either k_1 or k_2 is of this magnitude. Consequently the second term in eq. (3.11c) is negligible compared to the first term, and this happens in all higher orders of perturbation theory, whenever the projector P_0 appears. At large volumes, k_0 may be neglected everywhere compared to other momenta (see the appendix of ref. [4] for more details), with the result

$$\kappa_1 = -ik_{0\mu}\langle k_0, b|A_\mu|k_0', c\rangle, \tag{3.12a}$$

$$\kappa_2 = -(-ik_{0\mu})\langle 0, b|A_\mu K_0^{-1} A_\nu|0, c\rangle(-ik_{0\nu}'), \tag{3.12b}$$

$$\kappa_3 = (-ik_{0\mu})\langle 0, b|A_\mu K_0^{-1} K_1 K_0^{-1} A_\nu|0, c\rangle(-ik_{0\nu}'), \tag{3.12c}$$

etc. The whole perturbation series for κ may now be summed, and we obtain, with

$$\kappa_R \equiv \sum_{n=2}^{\infty} \kappa_n$$

$$\kappa_R = -(-ik_{0\mu})\langle 0, b|A_\mu K^{-1}[A] A_\nu|0, c\rangle(-ik_{0\nu}')$$

$$\kappa_R = -V^{-1}(-ik_{0\mu})\int d^D x\, d^D y\, A_\mu(x) K^{-1}(x, y; [A])\, A_\nu(y)(-ik_{0\nu}'). \tag{3.13}$$

Here color indices are suppressed, and $A_\mu(x)$, $A_\nu(y)$ and $K^{-1}(x, y; [A])$ are matrices in the adjoint representation of the Lie algebra of the gauge group. With this result, positivity of the Faddeev–Popov operator reduces to positivity of the $2DM$ dimensional matrix

$$\kappa = k_0^2 P_0 + \kappa_1 + \kappa_R. \tag{3.14}$$

The reader will recognize that expression (3.13) is correct in a scattering situation, but it is not correct for an evaluation of the lowest eigenvalue when a bound state is present. For an example where the result is not correct, suppose that a single component of $A_\mu(x)$ is slowly varying and of order C, throughout a euclidean volume U which is of linear dimension of order $R < L$. Then by choosing a trial wave function ψ supported in U, one obtains the estimates

$$(\psi, K_0\psi) \approx R^{-2}, \tag{3.15a}$$

$$(\psi, K_1\psi) \approx -R^{-1}|C|, \tag{3.15b}$$

$$(\psi, K\psi) \approx R^{-2} - R^{-1}|C|, \tag{3.15c}$$

where, in accordance with the uncertainty principle, $i\partial_\mu$ is taken to be of order R^{-1}. The color vector hidden in ψ is chosen to give the negative sign in eq. (3.15). If $|C|$ is sufficiently large, this expression will be arbitrarily large and negative. On the other hand, the formulas just obtained by summing perturbation theory, which give κ_0, κ_1 and κ_R of order k_0^2, $|k_0|$ and k_0^2 respectively, where $|k_0| = 2\pi/L$, were derived under the assumption of large euclidean volume, $L \to \infty$, so $k_0 \to 0$.

In the face of this difficulty, two remarks are in order. The first is that the error is an *overestimate* of the lowest eigenvalue $\lambda[A]$ when a bound state is present. Consequently the error made in the fundamental modular region, which is defined by the condition $\lambda[A] > 0$, is a fundamental modular region which is *too large*. Thus if a more refined treatment requires modification of our result, which is based on eq. (3.13), the correction will be a restriction to a fundamental modular region which is *smaller* than the one used here. In this case, the impact on the physics of the restriction to the fundamental modular region will be more drastic than what we shall find. The second remark is that the restriction to connections A for which the exact lowest eigenvalue $\lambda[A]$ is positive, eliminates those A which produce bound states, so the connections A for which the calculation (3.13) is incorrect are eliminated. Consequently, the restriction $\lambda[A] > 0$, with $\lambda[A]$ taken from eq. (3.13), may be sufficient after all to exclude those A for which the scattering calculation leading to eq. (3.13) is incorrect. This issue does not involve the intricacies of quantum field theory, and may be settled by an investigation of the Faddeev–Popov operator which is of Schrödinger type, generalized to include the color degree of freedom.

Likewise, we shall replace the condition that the least eigenvalue of the $2MD$-dimensional matrix $\kappa = \kappa[A]$ be positive by the weaker condition that $\operatorname{tr}\kappa[A]$ be positive. The above two remarks again apply: (1) The correct fundamental region may be smaller than the one obtained here, in which case the impact of the restriction to a fundamental modular region will be more drastic than what we find. (2) There may not be any error after all, because this condition may be strong enough to restrict the connections A to ones for which scattering theory does apply, in which case the $2MD$ least eigenvalues become degenerate as the euclidean volume $V \to \infty$. Again, this issue does not involve the intricacies of quantum field theory, but depends only on the properties of an operator of Schrödinger type with a color degree of freedom.

The replacement of the condition that the least eigenvalue of $\kappa[A]$ be positive by the condition that $\operatorname{tr}\kappa[A]$ be positive will allow an enormous simplification, because of the explicit expression, obtained from eqs. (3.13) and (3.14), with its simple volume dependence,

$$\operatorname{tr}\kappa = k_0^2\, 2DM - k_0^2\, 4CQ[A], \tag{3.16a}$$

where

$$Q[A] \equiv -(2CV)^{-1}\int \mathrm{d}^D x\, \mathrm{d}^D y\, \operatorname{tr}\left[A_\mu(x)\, K^{-1}(x, y; [A])\, A_\mu(y)\right] \tag{3.16b}$$

and tr means trace on the color indices. [For future convenience we have normalized Q with the factor C^{-1}, where C is the (positive) value of the Casimir operator in the adjoint representation $C\delta^{ab} = -f^{abc}f^{cda}$.] We have used the fact that κ_1 is traceless because

$$\langle k_0, b|A_\mu|k_0, b\rangle = V^{-1}\int \mathrm{d}^D x f^{bcb} A_\mu^c(x) = 0,$$

due to the anti-symmetry of the structure constants.

As a final comment we note that the positivity condition which we obtain at large volume, violates the scale and conformal invariance of the classical theory in 4 euclidean space-time dimensions. Of course it is not unusual for a gauge condition to violate some symmetry of the classical theory. In such a case, the quantum theory may or may not recover the classical symmetry for gauge-invariant observables. In 4 euclidean space-time dimensions it is well known that these symmetries are not regained. This has been generally thought to be caused by the introduction of an ultra-violet cut-off. However, in the present approach, the violation of scale and conformal invariance arises from a gauge-fixing procedure which is introduced in a finite euclidean box with periodic boundary conditions.

4. Non-local measure for non-abelian gauge theory

We now return to a consideration of the measure of non-abelian gauge theory given in eq. (3.1). It contains the factor $\theta(\lambda[A])$, which limits the integration of the Faddeev–Popov measure at the Gribov horizon. Here $\lambda[A]$ is the least eigenvalue of the $2DM$ dimensional matrix $\kappa[A]$ found in the last section, and, as discussed there, we shall replace $\theta(\lambda[A])$ by $\theta(\mathrm{tr}\,\kappa[A])$ given in eq. (3.16). With this substitution, the measure (3.1) may be expressed as

$$\mathrm{d}\mu_c \equiv \mathrm{d}A\,\delta(\partial\cdot A)\exp\left(-g^{-2}S_{\mathrm{cl}}\right)\det(K[A])\,\theta\left(c - g^{-2}Q[A]\right), \tag{4.1}$$

where

$$c = MD(2C)^{-1}g^{-2}, \tag{4.2}$$

and $Q[A]$ is given in eq. (3.16b). (The factors of g^{-2} are introduced for later convenience in a perturbative expansion.) In lowest order $K = K_0 = -\partial^2$, and the shape of the Gribov horizon precisely agrees with the model analyzed in sect. 2, where we have seen that the volume of the infinite-dimensional ellipsoid gets concentrated at its surface. Thus, in lowest order, it is correct to replace $\theta(c - g^{-2}Q[A])$ by $\delta(c - g^{-2}Q[A])$ as in eq. (2.10), and higher-order effects will implicitly be treated as perturbations of this lowest-order shape. Note that the convexity of Ω, mentioned in the introduction, and the rigorous ellipsoidal bound

(1.7) on Ω severely restrict its exact shape. [The Faddeev–Popov determinant actually vanishes on this boundary. However, this does not affect the conclusions of sect. 2, because the volume element in the radial variable r given by

$$\mathrm{d}r\, r^{N-1}(R-r)\,\theta(R-r)$$

also may be shown to approach const $\delta(R-r)$ as $N \to \infty$.] Thus we replace the measure (4.1) by

$$\mathrm{d}\mu_c = \mathrm{d}A\,\delta(\partial\cdot A)\exp\left(-g^{-2}S_{\mathrm{cl}}\right)\det(K[A])\,\delta\left(c - g^{-2}Q[A]\right). \tag{4.3}$$

Let us temporarily assume the equivalence of the microcanonical and canonical ensembles for this measure, so the δ-function may be replaced by the corresponding Boltzmann factor.

$$\mathrm{d}\mu_c = \mathrm{d}\mu_\gamma, \tag{4.4}$$

where

$$\mathrm{d}\mu_\gamma \equiv \mathrm{d}A\,\delta(\partial\cdot A)\exp\left[-g^{-2}(S_{\mathrm{cl}} + \gamma S_1[A])\right]\det(K[A]). \tag{4.5a}$$

Here $S_1 \equiv VQ[A]$, is given by

$$S_1 = -(2C)^{-1}\int \mathrm{d}^D x\,\mathrm{d}^D y\,\mathrm{tr}\left\{A_\mu(x)\,K^{-1}(x,y;[A])A_\mu(y)\right\}. \tag{4.5b}$$

(The minus sign appears because A, acting in the adjoint representation, is the color matrix $f^{abc}A^b$, which is anti-hermitian, so its square is negative.) By eq. (4.2) the value of the thermodynamic parameter γ is determined by the condition

$$MD(2C)^{-1}g^{-2} = c(\gamma) \equiv g^{-2}\langle Q[A]\rangle, \tag{4.6}$$

where the expectation value refers to the measure $\mathrm{d}\mu_\gamma$. By translational invariance, $c(\gamma)$ may be written

$$c(\gamma) = -g^{-2}(2C)^{-1}\int \mathrm{d}^D x\,\mathrm{tr}\left\langle A_\mu(x)K^{-1}(x,0;[A])A_\mu(0)\right\rangle. \tag{4.7a}$$

The thermodynamic parameter γ and the coupling constant g are related by

$$MD(2C)^{-1}g^{-2} = c(\gamma). \tag{4.7b}$$

This condition provides an absolute normalization for the connection A. It expresses the fact that the measure is supported at the boundary of the classical modular region.

To verify the equivalence of the microcanonical and canonical ensembles, we shall show that the variance of $Q[A]$ vanishes in the distribution $\mathrm{d}\mu_\gamma$. We have

$$\langle Q^2[A]\rangle = (2CV)^{-2}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w \Big\langle \mathrm{tr}\big[A_\mu(x)K^{-1}(x,y;A)A_\mu(y)\big] \times \mathrm{tr}\big[A_\nu(z)K^{-1}(z,w;A)A_\nu(w)\big]\Big\rangle ,$$

$$\langle Q^2[A]\rangle = (2C)^{-2}V^{-1}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \Big\langle \mathrm{tr}\big[A_\mu(x)K^{-1}(x,y;A)A_\mu(y)\big] \times \mathrm{tr}\big[A_\nu(z)K^{-1}(z,0;A)A_\nu(0)\big]\Big\rangle ,$$

by translation invariance, and we have written $\mathrm{d}x$ for the volume element d^Dx etc. By the cluster property at large volumes, we have, to leading order in the volume

$$\langle Q^2[A]\rangle = (2CV)^{-1}\int \mathrm{d}x\,\mathrm{d}y \Big\langle \mathrm{tr}\big[A_\mu(x)K^{-1}(x,y;A)A_\mu(y)\big]\Big\rangle \times (2C)^{-1}\int \mathrm{d}z \Big\langle \mathrm{tr}\big[A_\nu(z)K^{-1}(z,0;A)A_\nu(0)\big]\Big\rangle ,$$

which gives, by translation invariance,

$$\langle Q^2[A]\rangle = \langle Q[A]\rangle^2 .$$

Thus the variance of $Q[A]$ vanishes, so the random variable $Q[A]$ has a δ-function distribution in the measure $\mathrm{d}\mu_\gamma$, as required.

To see the significance of the new non-local term S_1 in the action (4.5), consider a point A in the interior of Ω, the region where the eigenvalues of $K[A]$ are all positive, and consider what happens as A approaches the boundary of Ω. The lowest eigenvalue $\lambda[A]$ of $K[A]$ approaches zero. Then, because K^{-1} appears in S_1, the probability gets very strongly suppressed by the factor $\exp(-\mathrm{const}/\lambda[A])$ as the boundary of Ω is approached.

The new term dominates the dynamics in the infrared region. In particular, it strongly suppresses the gluon propagator in the infrared region, causing the zero-order gluon propagator, (5.8) below, to vanish like k^2 at $k=0$. This reflects the proximity of the Gribov horizon in the directions (in A-space) of the long-wave-length components of A. The bound obtained in ref. [4] shows that in $D=4$ dimensions, the new term also causes enough suppression in the ultra-violet to contradict the predictions of the perturbative renormalization group.

5. Local action and conclusion

For many purposes it is convenient to have a local action, at the cost of introducing additional fields. This may be obtained starting with the identity

$$\exp(-g^{-2}\gamma S_1) = (\det K)^{MD/2} \times \int \mathrm{d}\varphi \exp\left\{-g^{-2}\int \mathrm{d}^D x\left[\tfrac{1}{2}\varphi_\mu^{ab}K^{bc}\varphi_\mu^{ac} + i\gamma^{1/2}C^{-1/2}f^{abc}\varphi_\mu^{ac}A_\mu^b\right]\right\}, \tag{5.1a}$$

which holds by gaussian integration. Here $\varphi_\mu^{ab}(x)$ is a real bose field, where μ runs from 1 to D, and a and b each run independently from 1 to M, where M is the dimension of the gauge group. Thus φ has DM^2 independent real components. If MD is even, then $(\det K)^{MD/2}$ may be represented by integration over $MD/2$ pairs of anti-commuting Faddeev–Popov ghosts. If MD is odd, one may introduce an additional odd number of the bose fields, coupled as in eq. (5.1a) but without the last term, or the result may be obtained by analytic continuation in MD, or D, from the even case. (All cases could be treated symmetrically by introducing MD complex bose fields and MD pairs of anti-commuting Faddeev–Popov ghosts.) For simplicity, we take MD even. To include also the factor $\det K$ which appears in eq. (4.5), we write

$$(\det K)^{(1+MD/2)} = \int \mathrm{d}C\,\mathrm{d}C^* \exp\left(-g^{-2}\int \mathrm{d}^D x\, C_i^* K C_i\right), \tag{5.1b}$$

where C_i and C_i^* are $(1+MD/2)$ pairs of independent Faddeev–Popov anti-commuting ghost fields, with ghost-flavor index $i = 1,\ldots,(1+MD/2)$. Finally, with

$$\delta(\partial\cdot A) = \int \mathrm{d}B \exp\left(-g^{-2}\int \mathrm{d}^D x\, iB\,\partial\cdot A\right), \tag{5.1c}$$

we put all the factors together, so the non-local measure $\mathrm{d}\mu_\gamma$, given in eq. (4.5), may be replaced by the measure

$$\mathrm{d}\mu = \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C\,\mathrm{d}C^*\,\mathrm{d}\varphi \exp(-g^{-2}S), \tag{5.2a}$$

where the local action S is defined by

$$S \equiv S_{\mathrm{cl}} + \int \mathrm{d}^D x\left[iB\,\partial\cdot A + C_i^* K C_i\right] + S_2, \tag{5.2b}$$

and

$$S_2 \equiv \int \mathrm{d}^D x\left(\tfrac{1}{2}\varphi_\mu^{ab}K^{bc}\varphi_\mu^{ac} + i\gamma^{1/2}C^{-1/2}f^{abc}\varphi_\mu^{ac}A_\mu^b\right). \tag{5.2c}$$

This action is renormalizable by power counting in $D=4$ euclidean space-time dimensions. In terms of local fields, the absolute normalization (4.7) of the connection may be written

$$MD(2C)^{-1} = i2^{-1}\gamma^{-1/2}\langle C^{-1/2} f^{abc} A_\mu^b \varphi_\mu^{ac} \rangle , \tag{5.3}$$

where the fields may be evaluated at, say, the origin, and the expectation value refers to the measure $d\mu$.

We rewrite our final formulas in a form suitable for perturbative calculations, which will enable us also to discuss its perturbative renormalizability, namely, to show that divergences which occur as $\epsilon = 4 - D$ approaches zero are of the same form as the terms present in the original action. For this purpose, we rescale the fields and actions according to

$$\begin{gathered} A = gA', \qquad F = gF', \\ B = gB', \qquad C_i = gC_i', \\ C_i^* = gC_i^{*\prime}, \qquad \varphi = g\varphi', \\ g^{-2}S[A, B, C, C^*, \varphi] = S'[A', B', C', C^{*\prime}, \varphi'], \\ g^{-2}S_{\mathrm{cl}}[A] = S_{\mathrm{cl}}'[A'], \\ g^{-2}S_2[A, \varphi] = S_2'[A', \varphi'], \qquad K[A] = K'[A'] . \end{gathered} \tag{5.4}$$

Whereas A is a geometrical quantity, the connection on a principle fiber bundle, the quantity A' which is introduced here is merely a calculational convenience for a perturbative expansion. We call it the perturbative gauge potential. (Had we quantized canonically, it would be the canonical field.) These quantities are given explicitly by

$$F_{\mu\nu}'^a = \partial_\mu A_\nu'^a - \partial_\nu A_\mu'^a + g f^{abc} A_\mu'^b A_\nu'^c , \tag{5.5a}$$

$$S_{\mathrm{cl}}'[A'] = \tfrac{1}{4} \sum_{\mu, \nu, a} \int d^D x \left(F_{\mu,\nu}'^a\right)^2(x) , \tag{5.5b}$$

$$K'[A'] = -\partial^2 - gA' \cdot \partial , \tag{5.5c}$$

$$\begin{aligned} S_2'[A', \varphi'] \equiv \int d^D x \Big[& \tfrac{1}{2} \varphi_\mu'^{ab} K'^{bc}(A') \varphi_\mu'^{ac} \\ & + i\gamma^{1/2} \varphi_\mu'^{ac} C^{-1/2} f^{abc} A_\mu'^b \Big] , \end{aligned} \tag{5.5d}$$

$$S' \equiv S_{\mathrm{cl}}' + \int d^D x \left[iB' \partial \cdot A' + C_i^{*\prime} K' C_i' \right] + S_2' , \tag{5.5e}$$

and the measure (5.2) may be expressed as

$$\mathrm{d}\mu = \mathrm{d}A'\,\mathrm{d}B'\,\mathrm{d}C'\,\mathrm{d}C^{*\prime}\,\mathrm{d}\varphi' \exp(-S'). \tag{5.6}$$

The coupling parameter $g = g(\gamma, D)$ is determined as a function of the thermodynamic parameter γ and the dimension D by elimination from

$$c(\gamma, g, D) = i2^{-1}\gamma^{-1/2}\langle C^{-1/2} f^{abc} A_\mu'^{b} \varphi_\mu'^{ac}\rangle, \tag{5.7a}$$

$$MD(2C)^{-1} g^{-2} = c(\gamma, g, D), \tag{5.7b}$$

where the expectation value refers to the measure (5.6).

We outline how calculations may be performed and divergences segregated. Basically two steps are involved. Step 1 is a perturbative calculation to given order in g, using the action (5.5). Step 2 is the non-perturbative normalization condition (5.7).

Step 1. Correlation functions in D euclidean space-time dimensions are calculated perturbatively in the coupling parameter g, using the action (5.5). Regularization is required, and is accomplished dimensionally, with $\epsilon = 4 - D$ as regulator. To zeroth order in g, the gluon propagator is given by

$$\left(\delta_{\mu\nu} - k_\mu k_\nu / k^2\right) \delta^{bc} \left[k^2 + \gamma/k^2\right]^{-1}, \tag{5.8a}$$

which may be re-expressed using

$$\left[k^2 + \gamma/k^2\right]^{-1} = \tfrac{1}{2}\left[\left(k^2 + i\gamma^{1/2}\right)^{-1} + \left(k^2 - i\gamma^{1/2}\right)^{-1}\right]. \tag{5.8b}$$

A calculation using this propagator is described in the appendix. The φ'–φ' propagator is given by

$$P\left(k^2 + \gamma/k^2\right)^{-1} + (I - P)\left(k^2\right)^{-1}, \tag{5.9a}$$

where P is the projector onto the transverse adjoint part of the φ' field, namely in momentum space

$$(P\varphi)_\mu^{ab} = C^{-1} f^{acb} f^{dce}\left(\delta_{\mu\nu} - k_\mu k_\nu / k^2\right)\varphi_\nu^{de}. \tag{5.9b}$$

The $A' - \varphi'$ propagator given by

$$i\gamma^{1/2} C^{-1/2} f^{acb}\left(\delta_{\mu\nu} - k_\mu k_\nu / k^2\right)\left[\left(k^2\right)^2 + \gamma\right]^{-1}, \tag{5.10}$$

and finally, the $C_i^{a\prime} - C_j^{b*\prime}$ propagator by

$$\delta_{ij}\delta^{ab}1/k^2, \tag{5.11}$$

where i and j run from 1 to $(1 + MD/2)$.

To show renormalizability, we use dimensional regularization to make all Feynman integrals finite. We must show that the divergences which appear as poles in $\epsilon \equiv 4 - D$ at $\epsilon = 0$ can be absorbed by renormalizing the constants and fields which occur in the action (5.5). This action contains the original Faddeev–Popov action and fields, enlarged by the action of $P = MD/2$ pairs of commuting ghosts φ', and $P = MD/2$ pairs of anti-commuting ghosts C' and $C^{*\prime}$. Suppose first that $\gamma^{1/2}$, which appears only in the term

$$i\gamma^{1/2}\varphi_\mu^{\prime ac}C^{-1/2}f^{abc}A_\mu^{\prime b}, \tag{5.12}$$

vanishes: $\gamma^{1/2} = 0$. Then the original Faddeev–Popov theory is recovered if the additional ghosts do not appear in external legs. For the $MD/2$ pairs of commuting ghosts φ' produce the factor $(\det K')^{-MD/2}$ which cancels the factor $(\det K')^{MD/2}$ produced by the additional $MD/2$ pairs of anti-commuting ghosts C' and $C^{*\prime}$. This is manifested in each order of perturbation theory by the appearance of the closed loops of the φ and of the C and C^* which are equal in value, but have opposite sign. To see what divergences occur in the individual Green functions in the enlarged theory with arbitrary kinds of external ghost legs (but still with $\gamma^{1/2} = 0$), we observe that each diagram of the enlarged theory, is equal, to within a sign, to a diagram of the unenlarged Faddeev–Popov theory. The only difference is that there are additional ghost labels to place on the same Feynman integrals, but their numerical value is the same (to within a sign), regardless of the ghost-flavor index, and regardless of whether a ghost is bosonic or fermionic. Therefore, no new divergences (i.e. poles in $\epsilon = 4 - D$) appear in the enlarged theory, for example as coefficients of φ^4, since there is no quartic ghost divergence in the original Faddeev–Popov theory. Similarly the poles that do appear as coefficients of the bilinear ghost–ghost terms or of the trilinear ghost–ghost-connection are the same in the enlarged theory as in the unenlarged theory, since the diagrams are equal in both theories, but differ only in their label. Thus, as long as $\gamma^{1/2} = 0$, the divergences can all be absorbed in a renormalization of the one coupling constant g, and of the fields, just as in the original Faddeev–Popov theory. Now consider what happens when $\gamma^{1/2} \neq 0$. This quantity appears only in the action in the off-diagonal $(\text{mass})^2$ term (5.12). Thus no new vertices appear. Only the zero-order propagators are changed. The A'–A' and φ'–φ' propagators are given in eqs. (5.8), and (5.9). At large momentum k, the non-vanishing of γ modifies these propagators only by terms of relative order γ/k^4. However, since there are no quartic divergences in the theory with $\gamma = 0$, this does not modify the divergence structure found previously. If these were the only propagators, we would be done. However, the off-diagonal A'–φ' propagator given in eq. (5.10) introduces new diagrams. At large momentum k, this propagator is of order $\gamma^{1/2}/k^4$, which is down by the factor $\gamma^{1/2}/k^2$ compared to a conventional propagator. Consequently there are no primitively divergent diagrams containing two or more propagators with high-momentum behavior $\gamma^{1/2}/k^4$, since at large k each of these propagators has two more powers of k in the denominator than conventional propagators. For the same reason, the only primitively divergent diagrams containing one propagator with high-momentum behavior $\gamma^{1/2}/k^4$ are self-energy diagrams, with a logarithmic (instead of quadratic) mass divergence. However all vertices contain an even number of Bose ghosts, so a self-energy diagram with one A'–φ' propagator must be an A'–φ' self-energy diagram. Therefore this new logarithmic mass divergence can be absorbed into a renormalization of $\gamma^{1/2}$ which is present only in the off-diagonal mass term (5.12). Thus it appears that all the divergences associated with the action (5.5) can be absorbed in a renormalization of the parameters and the fields which appear in it.

Step 2. The perturbative theory just described is not a gauge theory unless g has the particular value determined by the absolute normalization condition (5.7). To find g, the function $c(\gamma, g, D)$, given in eq. (5.7a), is calculated perturbatively, as a power series in the coupling constant g. Then $g = g(\gamma, D)$ is obtained by elimination from the non-perturbative relation (5.7) and substituted into the expression for the correlation function obtained in step 1. Although eq. (5.7b) is non-perturbative in g, as witnessed by the factor g^{-2} on the left, nevertheless it may be solved iteratively in g. To do so, we regard $c(\gamma, g, D)$, given in eq. (5.7a) as a power series in g. Then, starting with the zeroth-order expression for $c(\gamma, g, D)$, we consistently eliminate g from the right-hand side of eq. (5.7b), up to any given power of g. From the zeroth-order expression for $c(\gamma, g, D)$ we obtain

$$MDg^{-2} = (2\pi)^{-D} CM(D-1) \int \mathrm{d}^D k \left[(k^2)^2 + \gamma \right]^{-1}, \tag{5.13}$$

where $M = \sum_b \delta^{bb}$, is the dimension of the adjoint representation. The integral over k is given in eq. (2.24), with the result

$$g^{-2} = 2[\Gamma(D/2)]^{-1}(4\pi)^{-D/2}(\pi/4)[\sin(\pi D/4)]^{-1} C(D-1) D^{-1} \gamma^{-1+D/4}. \tag{5.14}$$

We find for small $\epsilon = 4 - D$

$$g^{-2} \approx (4\pi)^{-2}(3/2) C \left[\epsilon^{-1} - \ln(\gamma^{1/4}/\text{const}) \right]. \tag{5.15}$$

The unrenormalized coupling constant g^2 vanishes with ϵ, so this relation expresses asymptotic freedom. It is remarkable that it is obtained here in zeroth order, without evaluating any Feynman diagrams, or doing a resummation of perturbation series, as in the perturbative renormalization group. The coefficient obtained here differs from that of the conventional perturbative renormalization group, but contributions from higher-order terms must be estimated before a comparison is made.

The geometrical significance of the absolute normalization condition (5.3) or (5.7), is that in the functional integral, the connection $A = gA'$ which is a geometrical quantity, ranges over the classical configuration space, which is bounded by the Gribov horizon. Consequently there is no renormalization of A. Its absolute normalization expresses the fact that the functional integral over connections extends precisely over the *classical* configuration space, in accordance with Feynman's conception of quantization as a sum over *classical* paths. Physical quantities are gauge-invariant functions of the geometrical connection A, and we should expect of a satisfactory theory, that a sufficiently large class of gauge-invariant functions of A such as the Wilson loops has finite expectation value.

For $D < 4$, the absolute normalization condition (5.7) exchanges the perturbative dimensionful coupling parameter g for the non-perturbative dimensionful parameter γ. Continued to $D = 4$ dimensions (after a resummation of the perturbation series), the absolute normalization condition provides an explicit expression of the phenomenon of dimensional transmutation, in which the dimensionless perturbative coupling parameter g is replaced by the non-perturbative dimensionful parameter γ. As in lattice gauge theory, no renormalized coupling constant enters the theory, and g corresponds to the unrenormalized coupling constant in a non-gauge theory. The dimensional regularization which is required for $D = 4$ dimensions, whereby $g = g(\epsilon)$ is a function of $\epsilon = 4 - D$, is the analog of the dependence of $g = g(a)$ on the lattice spacing a, which is present in lattice gauge theory in $D = 4$ dimensions.

Our discussion so far has dealt with the euclidean theory. However for a physical interpretation, it is necessary that the minkowskian correlation functions, obtained by analytic continuation from the euclidean ones, possess the positivity properties which imply a physical Hilbert space with a positive metric. In particular, physical propagators must satisfy the Kallen–Lehmann representation

$$g(k) = \int_0^\infty \mathrm{d}m^2 \rho(m^2)/(k^2 + m^2), \qquad \rho \geqslant 0. \tag{5.16a}$$

At large k this gives the $1/k^2$ behavior

$$g(k) \approx \int_0^\infty \mathrm{d}m^2 \rho(m^2)/k^2, \qquad \rho \geqslant 0 \tag{5.16b}$$

if this integral converges, and a less rapid fall-off at large k if it does not. Either behavior violates the rigorous bound (1.8) in $D=4$ euclidean space-time dimensions, which requires that $g(k)$ falls off more rapidly than $1/k^2$. Thus the propagator of the connection cannot satisfy the Kallen–Lehmann representation (unless ρ vanishes identically), and we should not be surprised that the free propagator of the connection A, (5.8), possesses poles at $p^2 = \pm i\gamma^{1/2}$ which are clearly incompatible with this representation. However A is not a gauge-invariant field, and so it need not satisfy the Kallen–Lehmann representation.

To gain further insight, we calculate in zeroth order the correlation function of two small Wilson loops which, in a gauge theory, are the natural candidates for producing localized physical excitations. The details of the calculation of its Fourier transform $G(p)$ are given in the appendix. The result is that $G(p)$ may be written as the sum of two terms

$$G(p) = G^{\mathrm{un}}(p) + G^{\mathrm{ph}}(p), \tag{5.17}$$

with the following properties. The singularities of $G^{\mathrm{un}}(p)$ are the two-gluon cuts beginning at the unphysical values $p^2 = \pm 4i\gamma^{1/2}$ which, in virtue of the Landau–Cutkosky rules, are an inevitable consequence of the unphysical gluon poles at $p^2 = \pm i\gamma^{1/2}$. On the other hand, $G^{\mathrm{ph}}(p)$ has a cut which begins at the physical threshold $p^2 = -2\gamma^{1/2}$, and the corresponding spectral function is positive. Thus $G^{\mathrm{ph}}(p)$ does satisfy the Kallen–Lehmann representation, and hence it is an acceptable correlation function for physical excitations and is fit to describe the propagation of glueballs.

What are we to make of this situation? Note that in a confining theory, *any* approximation scheme which starts with a free gluon field *must* have the feature that poles (or some other kind of singularity) are present in the zeroth-order gluon propagator, but absent from the exact propagator. Inevitably, the unphysical poles present in the zero-order gluon propagator produce unphysical cuts also in correlation functions of gauge-invariant quantities, when these are calculated perturbatively (in addition to the physical cuts we have found). However, the unphysical cuts cannot be present in the exact correlation functions of gauge-invariant quantities. Therefore, as higher-order contributions are included, the unphysical cuts must leave, for example by retreating to infinity if the theory is confining, or by moving back to the real axis and becoming physical if it is not. The fact that in zeroth order the gluon singularities are at an unphysical location, but glueball correlation functions nevertheless do have physical singularities, suggests simply that gluons are not physical excitations, but glueballs are.

The shape of the classical configuration space has, in zeroth order, moved the gluon poles to an unphysical location where, by mere consistency, they cannot be tolerated in the exact theory. Thus it would appear that the shape of the classical configuration space is directly responsible for the confinement of gluons. This proposed confinement mechanism is kinematic rather than dynamic, and is sufficiently simple and direct, appearing already in zeroth order, that it is to be expected that it will survive possible further refinement of the theory.

It appears that the zeroth-order approximation to the present theory accords with our general ideas about QCD. In $D=4$ dimensions, the zeroth-order coupling constant $g(\epsilon)$ vanishes as $\epsilon = 4 - D$ approaches zero, as is characteristic of asymptotically free theories. The singularities associated with gluons occur at unphysical mass, whereas singularities in the glueball channel occur at a physical mass and satisfy physical positivity. We interpret this as an indication of confinement in the exact theory. Although we have made a number of unsubstantiated hypotheses in the derivation, the error introduced, if any, is, in each case, to *increase* the size of the fundamental modular region. Thus, if a more refined treatment is called for, the impact of the restriction to the fundamental modular region will be more drastic than the one found here. On the other hand, the result obtained here would be favored if nature prefers simplicity; for a local, renormalizable theory is simpler than one would expect to result from the imposition of a global restriction on the classical configuration space. We have discussed only the application of the present theory to QCD. However the restriction to a fundamental modular region must also be imposed in the gauge theory of the electro-weak interactions. Here too, it is possible that the appearance of the dimensionful parameter γ which this restriction induces may be relevant for the generation of mass.

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Appendix

We are interested in the spectral properties of gauge-invariant quantities, with particular attention to their positivity properties in the minkowskian region. In the absence of a non-perturbative analysis we present the calculation of the connected correlation function of two small Wilson loops centered at x and at the origin, respectively,

$$\langle W(x) W(0) \rangle_{\mathrm{c}}, \tag{A.1}$$

to zeroth order of perturbation theory. Each loop is of small spatial extent and encloses a small area element $\Sigma_{\mu\nu}$. Under these conditions, the above expression reduces, within a multiplicative constant, to

$$\left\langle (\Sigma_{\mu\nu} F_{\mu\nu})^2(x) (\Sigma_{\kappa\lambda} F_{\kappa\lambda})^2(0) \right\rangle_{\mathrm{c}}. \tag{A.2}$$

We orient the coordinate axes, so the area element $\Sigma_{\mu\nu}$ lies in the 1–2 plane, and it is sufficient to consider

$$\left\langle F_{12}^2(x) F_{12}^2(0)\right\rangle_{\mathrm{c}}, \tag{A.3}$$

which is the propagator of the field F_{12}^2. To lowest order in g, $F_{12}=\partial_1 A_2-\partial_2 A_1$, and we obtain to zeroth order,

$$2 \sum_{ab}\left[\left\langle\left(\partial_1 A_2-\partial_2 A_1\right)^a(x)\left(\partial_1 A_2-\partial_2 A_1\right)^b(0)\right\rangle\right]^2 .$$

With $A=g A'$, where A' is the perturbative gauge potential, we obtain, again in zeroth order,

$$2 g^4 M\left[(2 \pi)^{-D} \int \mathrm{d}^D k \exp (i k \cdot x)\left(k_1^2+k_2^2\right)\left(k^2+\gamma / k^2\right)^{-1}\right]^2, \tag{A.4}$$

where the free gluon propagator (5.8) has made its appearance. With

$$\left\langle F_{12}^2(x) F_{12}^2(0)\right\rangle_{\mathrm{c}}=(2 \pi)^{-D} \int \mathrm{d}^D p \exp (i p \cdot x)\, G(p), \tag{A.5}$$

we obtain for the momentum-space propagator

$$\begin{aligned} G(p)= & 2 g^4 M(2 \pi)^{-D} \int \mathrm{d}^D k\left(k_1^2+k_2^2\right)\left[(p-k)_1^2+(p-k)_2^2\right] \\ & \times\left(k^2+\gamma / k^2\right)^{-1}\left[(p-k)^2+\gamma /(p-k)^2\right]^{-1} . \end{aligned} \tag{A.6}$$

To evaluate this integral, it is convenient to represent each propagator as

$$\left(k^2+\gamma / k^2\right)^{-1}=\tfrac{1}{2}\left[\left(k^2+i \gamma^{1/2}\right)^{-1}+\left(k^2-i \gamma^{1/2}\right)^{-1}\right]$$

$$\left(k^2+\gamma / k^2\right)^{-1}=\int_0^{\infty} \mathrm{d} \alpha \cos \left(\gamma^{1/2} \alpha\right) \exp \left(-k^2 \alpha\right), \tag{A.7}$$

and we have

$$\begin{aligned} G(p)= & 2 g^4 M \int_0^{\infty} \mathrm{d} \alpha \int_0^{\infty} \mathrm{d} \beta \cos \left(\gamma^{1/2} \alpha\right) \cos \left(\gamma^{1/2} \beta\right) \\ & \times(2 \pi)^{-D} \int \mathrm{d}^D k\left(k_1^2+k_2^2\right)\left[(p-k)_1^2+(p-k)_2^2\right] \exp \left[-k^2 \alpha-(p-k)^2 \beta)\right] \end{aligned} \tag{A.8}$$

The integral over k may be performed, after a shift, by the usual gaussian integration, with the result

$$G(p) = g^4 M \int_0^\infty \mathrm{d}\alpha \int_0^\infty \mathrm{d}\beta \left\{ \cos\left[\gamma^{1/2}(\alpha+\beta)\right] + \cos\left[\gamma^{1/2}(\alpha-\beta)\right] \right\}$$
$$\times \left[4\pi(\alpha+\beta)\right]^{-D/2} \exp\left[-\alpha\beta(\alpha+\beta)^{-1} p^2\right] Q, \qquad \text{(A.9a)}$$

where

$$Q \equiv \alpha^2\beta^2(\alpha+\beta)^{-4}\left(p_1^2+p_2^2\right)^2 + 2(\alpha-\beta)^2(\alpha+\beta)^{-3}\left(p_1^2+p_2^2\right) + 8(\alpha+\beta)^{-2}. \qquad \text{(A.9b)}$$

In the integrand we write

$$1 = \int_0^\infty \mathrm{d}\lambda\, \delta(\alpha+\beta-\lambda),$$

and change variables according to $\alpha = \lambda\alpha'$ and $\beta = \lambda\beta'$. The integration on λ is performed with the result, after dropping primes,

$$G(p) = g^4 M (4\pi)^{-D/2} \mathrm{Re}\left[\int_0^1 \mathrm{d}\alpha \int_0^1 \mathrm{d}\beta\, \delta(\alpha+\beta-1)(R_1+R_2+R_3)\right], \qquad \text{(A.10)}$$

where Re is the real part of the following expression, and

$$R_1 \equiv \Gamma(2-D/2)\alpha^2\beta^2\left(p_1^2+p_2^2\right)^2$$
$$\times \left\{\left(\alpha\beta p^2 - i\gamma^{1/2}\right)^{-2+D/2} + \left[\alpha\beta p^2 - i\gamma^{1/2}(\alpha-\beta)\right]^{-2+D/2}\right\}, \qquad \text{(A.11a)}$$

$$R_2 \equiv \Gamma(1-D/2)(\alpha-\beta)^2\left(p_1^2+p_2^2\right)$$
$$\times \left\{\left(\alpha\beta p^2 - i\gamma^{1/2}\right)^{-1+D/2} + \left[\alpha\beta p^2 - i\gamma^{1/2}(\alpha-\beta)\right]^{-1+D/2}\right\}, \qquad \text{(A.11b)}$$

$$R_3 \equiv 2\Gamma(-D/2)\left\{\left(\alpha\beta p^2 - i\gamma^{1/2}\right)^{D/2} + \left[\alpha\beta p^2 - i\gamma^{1/2}(\alpha-\beta)\right]^{D/2}\right\}. \qquad \text{(A.11c)}$$

Integration on β yields

$$G(p) = g^4 M (4\pi)^{-D/2} \int_0^1 \mathrm{d}\alpha\, (R_1+R_2+R_3)|_{\beta=1-\alpha}. \qquad \text{(A.12)}$$

The spectrum of intermediate states is found from the singularities in the complex p^2-plane. By writing

$$\mathrm{Re}\left[\alpha(1-\alpha)p^2 - i\gamma^{1/2}\right]^x = \tfrac{1}{2}\left[\alpha(1-\alpha)\right]^x\left\{\left[p^2 - i\gamma^{1/2}\alpha^{-1}(1-\alpha)^{-1}\right]^x + \left[p^2 + i\gamma^{1/2}\alpha^{-1}(1-\alpha)^{-1}\right]^x\right\}, \tag{A.13}$$

one sees from eq. (A.11) that the first term in R_1, R_2 and R_3 yields a contribution to $G(p)$ which is analytic in the cut p^2-plane, with cuts starting at $p^2 = \pm 4i\gamma^{1/2}$ that may be drawn along the imaginary axes on the segments

$$p^2 = 4i\gamma^{1/2} \quad \text{to} \quad i\infty, \qquad p^2 = -4i^{1/2} \quad \text{to} \quad -i\infty. \tag{A.14a, b}$$

The presence of singularities at $p^2 = \pm 4i\gamma^{1/2}$ could have been inferred from a Landau–Cutkosky analysis of the Feynman integral (A.6), which has a pinching of the singularities at $(k-p)^2 = \pm i\gamma^{1/2}$ and at $k^2 = \pm i\gamma^{1/2}$. These correspond to gluon masses at

$$m = \exp(\pm i\pi/4)\gamma^{1/4}, \tag{A.15}$$

giving $(2m)^2 = \pm 4i\gamma^{1/2}$. A singularity at imaginary p^2 and its associated cut are clearly unphysical and should not be present in the propagator of a gauge-invariant field in an exact physical theory. Unless there is an error in our quantization procedure, we can only suppose that these non-physical singularities will move out to infinity (or that their strength approaches zero) in an exact evaluation of this propagator, and that their appearance at imaginary p^2 results from the crudeness of the zeroth-order approximation. This can only happen if the pole in the gluon propagator moves to infinity (or its residue approaches zero) when higher-order effects are included, which would describe confinement of gluons. We call $G^{\mathrm{un}}(p)$ the total contribution to $G(p)$ which comes from the first term in R_1, R_2 and R_3.

On the other hand, the gluon masses given in eq. (A.15) by $m_{\pm} = \exp(\pm i\pi/4)\gamma^{1/4}$ may also pinch in the Feynman integral (A.6) and cause a singularity in the complex p^2-plane at

$$-p^2 \equiv (m_+ + m_-)^2 = \left[\exp(+i\pi/4)\gamma^{1/4} + \exp(-i\pi/4)\gamma^{1/4}\right]^2,$$
$$-p^2 = 2\gamma^{1/2}. \tag{A.16}$$

This does correspond to a physical spectrum. The second term in R_1, R_2 and R_3, has a cut here, although this is not yet obvious from the identities (A.11). We call the $G^{\text{ph}}(p)$ the contribution to $G(p)$ from the second term in eqs. (A.11a)–(A.11c), so

$$G(p) = G^{\text{ph}}(p) + G^{\text{un}}(p). \tag{A.17}$$

We evaluate $G^{\text{ph}}(p)$ to display this cut. For this purpose we change variables from α to

$$s \equiv \tfrac{1}{2}(\alpha - \beta)\alpha^{-1}\beta^{-1} = \tfrac{1}{2}(2\alpha - 1)\alpha^{-1}(1-\alpha)^{-1},$$

and obtain

$$G^{\text{ph}}(p) = g^4 M (4\pi)^{-D/2} \tfrac{1}{2} \int_{-\infty}^{+\infty} \mathrm{d}s\, (1+s^2)^{-1/2} \times \left[(1+s^2)^{1/2} + 1\right]^{-1} (R_1 + R_2 + R_3), \tag{A.18}$$

where

$$R_1 \equiv (p_1^2 + p_2^2)^2 \Gamma(2 - D/2) \left\{2\left[(1+s^2)^{1/2} + 1\right]\right\}^{-D/2} \left[p^2 - 2i\gamma^{1/2}s\right]^{-2+D/2}, \tag{A.19a}$$

$$R_2 \equiv (p_1^2 + p_2^2) \Gamma(1 - D/2)(4s^2) \left\{2\left[(1+s^2)^{1/2} + 1\right]\right\}^{-1-D/2} \left[p^2 - 2i\gamma^{1/2}s\right]^{-1+D/2}, \tag{A.19b}$$

$$R_3 \equiv 2\Gamma(-D/2) \left\{2\left[(1+s^2)^{1/2} + 1\right]\right\}^{-D/2} \left[p^2 - 2i\gamma^{1/2}s\right]^{D/2}. \tag{A.19c}$$

The contour of integration may be deformed from the real s-axis into the upper half of the complex s-plane. The only singularity encountered there is the square-root branch cut at $s = i$. By deforming the integral over s to surround the attached cut which may be drawn from $s = i$ to $s = i\infty$, one expresses the integral over s in terms of the discontinuity at the cut. With $s = iy$, one obtains

$$G^{\text{ph}}(p) = g^4 M (8\pi)^{-D/2} \int_1^{+\infty} \mathrm{d}y \,(y^2 - 1)^{-1/2} (T_1 + T_2 + T_3), \tag{A.20}$$

where

$$T_1 \equiv \left(p_1^2+p_2^2\right)^2 \Gamma(2-D/2)\, y^{-1-D/2} \cos[(1+D/2)\theta]\left[p^2+2\gamma^{1/2} y\right]^{-2+D/2}, \tag{A.21a}$$

$$T_2 \equiv -2\left(p_1^2+p_2^2\right) \Gamma(1-D/2)\, y^{-D/2} \cos[(2+D/2)\theta]\left[p^2+2\gamma^{1/2} y\right]^{-1+D/2}, \tag{A.21b}$$

$$T_3 \equiv 2\Gamma(-D/2)\, y^{-1-D/2} \cos[(1+D/2)\theta]\left[p^2+2\gamma^{1/2} y\right]^{D/2}, \tag{A.21c}$$

and

$$\cos\theta \equiv 1/y, \qquad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{A.22}$$

With this representation of $G^{\mathrm{ph}}(p)$, one sees that it has a cut in the complex p^2-plane, starting at $p^2=-2\gamma^{1/2}$ and extending along the negative real axis to $-\infty$. This is indeed a physical spectrum. However for a physical interpretation, it is also necessary that the corresponding states be part of a (positive metric) Hilbert space, which implies that the discontinuity at this cut be positive. To investigate this point, we use Cauchy's integral theorem to obtain a representation of a function which is analytic in the cut plane in terms of its discontinuity along the cut. In particular, one has

$$\left(p^2+t\right)^x=-\pi^{-1} \sin(\pi x) \int_t^{\infty} \mathrm{d}\tau \left(\tau+p^2\right)^{-1}(\tau-t)^x, \tag{A.23}$$

where t is real and positive, and x is real in the interval $-1<x<0$. With this representation and

$$\pi^{-1} \sin(-\pi x)=[\Gamma(-x) \Gamma(1+x)]^{-1}, \tag{A.24}$$

one obtains

$$G^{\mathrm{ph}}(p)=\int_\sigma^{+\infty} \mathrm{d}\tau\left(\tau+p^2\right)^{-1} \rho(\tau), \tag{A.25a}$$

where

$$\sigma \equiv 2\gamma^{1/2}. \tag{A.25b}$$

The spectral function is given by

$$\rho(\tau)=g^4 M(8\pi)^{-D/2} \int_0^{\varphi} \mathrm{d}\theta\left(U_1+U_2+U_3\right), \tag{A.26}$$

where

$$U_1 \equiv (p_1^2+p_2^2)^2 \Gamma^{-1}(-1+D/2)\tau^{-2+D/2}\cos^2\theta$$
$$\times \cos[(1+D/2)\theta](\cos\theta-\cos\varphi)^{-2+D/2}, \tag{A.27a}$$

$$U_2 \equiv -2(p_1^2+p_2^2)\Gamma^{-1}(D/2)\tau^{-1+D/2}\cos[(2+D/2)\theta](\cos\theta-\cos\varphi)^{-1+D/2}, \tag{A.27b}$$

$$U_3 \equiv 2\Gamma^{-1}(1+D/2)\tau^{D/2}\cos[(1+D/2)\theta](\cos\theta-\cos\varphi)^{D/2}. \tag{A.27c}$$

Here we have changed variable of integration from y to θ given in eq. (A.22), and $\varphi=\varphi(\tau)$ is defined by

$$\cos\varphi(\tau)=2\gamma^{1/2}/\tau, \qquad 0 \leqslant \varphi \leqslant \pi/2. \tag{A.28}$$

In general eq. (A.25) may require subtractions for convergence, so it is symbolic. However the spectral function does not contain any of the divergences of the original Feynman integral (A.6) so it is finite for arbitrary D, and is given correctly in eq. (A.26).

Physical positivity will be satisfied for $G^{\mathrm{ph}}(p)$ provided that $\rho(\tau)$ is positive. One may verify that it is indeed positive near the threshold at $\tau=2\gamma^{1/2}$, but the integrand changes sign so it is not obvious whether positivity is satisfied for all τ. To investigate this point we consider the cases $D=2$ and $D=4$, in which the integral may be done exactly. For $D=2$, we have $p_1^2+p_2^2=p^2$, and the spectral function in the minkowskian region is obtained by replacing p^2 by $-\tau$, as one sees from eq. (A.25). The result, for $D=2$ is

$$\rho_2(\tau)=g^4M(8\pi)^{-1}2\gamma^{1/2}\sin(2\varphi), \tag{A.29a}$$

$$\rho_2(\tau)=g^4M\pi^{-1}\gamma\left[\tau^2-4\gamma\right]^{1/2}\tau^{-2}, \tag{A.29b}$$

which is clearly positive. For $D=4$, one obtains

$$\rho_2(\tau)=g^4M(8\pi)^{-2}(15)^{-1}\sin\varphi V, \tag{A.30a}$$

where

$$V=2\left[\tau(1-\cos^2\varphi)+\tfrac{1}{2}(p_1^2+p_2^2)(1-6\cos^2\varphi)\right]^2$$
$$+\tfrac{1}{2}(p_1^2+p_2^2)\left[3+2\cos^2\varphi+12\cos^2\varphi(1-\cos^2\varphi)\right], \tag{A.30b}$$

and φ is defined in terms of τ in eq. (A.28). There are two cases to be considered. If p_1 and p_2 are space-like components, then this expression gives the spectral function directly in the minkowskian region. On the other hand, if either p_1 or p_2 are time-like, then the spectral function in the minkowskian region is obtained by writing

$$p_1^2 + p_2^2 = p^2 - p_3^2 - p_4^2 \to -\tau - p_3^2 - p_4^2\,, \qquad \text{(A.31)}$$

as one sees from eq. (A.25). Now p_3 and p_4 are space-like components, and the spectral function is the minkowskian spectral function. In either case, positivity is obvious from eq. (A.30).

The conclusion is that, in zeroth order, the propagator $G(p)$ of the gauge-invariant field $F_{12}^2(x)$ has two unphysical cuts beginning at $p^2 = \pm 4i\gamma^{1/2}$, associated with the unphysical location of the zero-order gluon poles $\pm i\gamma^{1/2}$, and a physical cut beginning at $p^2 = -2\gamma^{1/2}$, where physical positivity is satisfied. The interpretation of these results is discussed at the end of sect. 5.

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